SPLITTING OF SINGULARITIES

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Abstract. We study one parameter deformations of a pair consisting of an analytic singular space X_0 and a function f_0 on it, in case this defines an isolated singularity. We prove, under general conditions, a bouquet decomposition of the Milnor fibre when the isolated singularity splits in the deformation and the invariance of the Milnor fibration if there is no splitting.

1. Introduction

Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ be an analytic function germ defined on an analytic space germ $(X, 0)$ embedded in $(\mathbb{C}^{m+1}, 0)$. Let l be a linear function on $\mathbb{C}^{m+1}$, which is considered as the last coordinate function of $\mathbb{C}^{m+1}$. Let Δ be a small open disc in $\mathbb{C}$ with center 0, and U be a small open neighborhood of 0 in $\mathbb{C}^m$, such that in $W := U \times \Delta$, $(X, 0)$ can be represented as an analytic set. For each $t \in \Delta$, define $X_t := W \cap X \cap l^{-1}(t)$ and $f_t := f(-, t) = f|X_t$. Assume that X_t is irreducible and $f(0, t) = 0$ for any $t \in \Delta$. The triple (X, f, l), or briefly, the pair (X_t, f_t), is called *a one-parameter deformation of the (space-function) pair* (X_0, f_0).

Let $\mathcal{S} = \{\mathcal{S}_i\}$ be a Whitney stratification of X, the representative of $(X, 0)$ in W. Denote by $\Sigma_{\mathcal{S}}(f, l)$ the critical set of the mapping $(f, l) : X \longrightarrow \mathbb{C}^2$ with respect to the stratification $\mathcal{S}$. We study deformations of isolated singularities, defined as follows.

1.1. **Definition.** The triple (X, f, l) (or the pair (X_t, f_t)) is called a *one-parameter deformation of an isolated singularity* (X_0, f_0) if the intersection of $\Sigma_{\mathcal{S}}(f, l)$ with $l^{-1}(0)$ has the origin as an isolated point.

If (X_t, f_t) is a one-parameter deformation of an isolated singularity (X_0, f_0), then it follows that the dimension of $\Sigma_{\mathcal{S}}(f, l)$ is at most one and the intersection of $\Sigma_{\mathcal{S}}(f, l)$ with $l^{-1}(t)$ is of dimension 0 (or void) for small t. Since $\Sigma_{\mathcal{S}} l \subset \Sigma_{\mathcal{S}}(f, l)$, it follows that $l^{-1}(t)$ cuts transversally the positive dimensional strata of $\mathcal{S}$, except at a finite number of points, namely the points of the set $X_t \cap \Sigma_{\mathcal{S}} l$. By the transversality result of Cheniot [4], the stratification $\mathcal{S}_t$ of X_t which consists of $S_i \cap l^{-1}(t) \setminus \Sigma_{\mathcal{S}} l$ and the points $l^{-1}(t) \cap \Sigma_{\mathcal{S}} l$ is Whitney. Now the function $f_t : X_t \longrightarrow \mathbb{C}$ has at most isolated singularities in U with respect to the Whitney stratification $\mathcal{S}_t$ of X_t. The critical set of f_t is $\Sigma_{\mathcal{S}_t}(f_t) = l^{-1}(t) \cap \Sigma_{\mathcal{S}}(f, l)$.

1.2. **Definition.** If $\Sigma_{\mathcal{S}_t}(f_t)$ has only one point in U for small enough U and $|t|$, we say that the singularity (X_0, f_0) *does not split.*

In this case, it follows that the singular locus $\Sigma_{\mathcal{S}}(f, l)$ is non-singular, hence a line up to analytic change of coordinates.

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The following questions may arise in this context.

Conjecture A. The Milnor fibre of an isolated singularity (X_0, f_0) is homotopy equivalent to the bouquet of the Milnor fibres of the isolated singularities into which it splits.

Conjecture B. If the isolated singularity (X_0, f_0) does not split, then the Milnor fibration of the isolated singularity of (X_t, f_t) is homotopically constant, for t close to 0.

For the existence of the Milnor fibrations and the topology of the Milnor fibre we refer the reader to the papers of Lê [12, 13]. There is evidence for these statements as follows. Conjecture A holds when X_0 is a regular space. More generally, it holds when $X_t = X_0$, $\forall t$, X_0 has an isolated singularity, $\dim X_0 \neq 3$ and the singularity of f_0 splits such that outside the origin there are only Morse singularities, see Siersma's paper [21].

We show here that Conjecture A holds in homology (with any coefficients), for the most general setting. It then follows, by Whitehead's theorem, that it holds in homotopy when the singularity splits into only singular points whose local Milnor fibres are simply connected.

Conjecture B is an extension of a well known result of Lê and Ramanujam [14] in case $X = \mathbb{C}^m \times \mathbb{C}$. The Lê-Ramanujam result has been extended in another direction by Vannier [26, 27] and Massey [16], [17], in case $X = \mathbb{C}^m \times \mathbb{C}$ and f_t with non-isolated singularities on $\mathbb{C}^m$. Let us mention that in the classical case $X = \mathbb{C}^m \times \mathbb{C}$ and f_t with an isolated singularity, Timourian [25] proved furthermore that the right-equivalence class of f_t is constant.

With the usual restriction on dimension, we show that Conjectures A and B hold when for each small t the space X_t has isolated singularity and "link stability".

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2. Bouquet decompositions

The purpose of this section is to show that Conjecture A holds in homology in general and in homotopy under some conditions. We first extend a result of Siersma [21] on generic splittings to the case of any splitting.

2.1. Let Y be an analytic space of pure dimension $n+1$, locally embedded in $\mathbb{C}^m$ in a neighborhood of 0. Let $\mathcal{S}$ be a Whitney stratification of a representative of Y. Denote by B the open ball in $\mathbb{C}^m$ with radius ε and center 0, by D the open disk in $\mathbb{C}$ with radius η and center 0.

Let $f : Y \longrightarrow \mathbb{C}$ be an analytic function with isolated singularities in $B \cap Y$ with respect to the stratification $\mathcal{S}$ in the sense of [13]. Let $\Sigma(f) = \{P_0 = 0, P_1, \dots, P_\sigma\}$ be the critical set of f on $B \cap Y \cap f^{-1}(D)$. Denote by $b_i = f(P_i)$ $(i = 0, 1, \dots, \sigma)$. Assume that $\{P_0 = 0, P_1, \dots, P_\varsigma\} \subset Y_{\text{sing}} \neq \emptyset$, and $\{P_{\varsigma+1}, \dots, P_\sigma\} \subset Y_{\text{reg}}$. Moreover, we assume:

(∗) For all $u \in \bar{D}$, $(Y \cap f^{-1}(u)) \bar{\pitchfork} \partial \bar{B}$ (as stratified sets).

In B (resp. D), take a small closed ball $\bar{B}^i$ (resp. disc $\bar{D}^i$) around each P_i (resp. b_i) such that the restriction of f to $\bar{B}^i \cap Y \cap f^{-1}(\bar{D}^i \setminus \{b_i\})$ induces the local Milnor fibration. Let c_i $(1 \leq i \leq \sigma)$ be the path in $D \setminus \cup_{j=0}^{\sigma} \mathrm{int} \bar{D}^j$ connecting $u_0 \in \partial \bar{D}^0$ with $u_i \in \partial \bar{D}^i$ such that each path has no self-intersection and two paths intersect only at u_0. Without loss of generality, we assume that, if $b_i = b_j$ (resp. $b_i = b_0$), then $\bar{D}_i = \bar{D}_j$ and $c_i = c_j$ (resp. c_i is the constant path at u_0). For any $A \subset \mathbb{C}$, denote $Y_A := Y \cap \bar{B} \cap f^{-1}(A)$. Set

$$E := Y_{\bar{D}}, \quad \hat{F} := Y_{u_0}, \qquad E^i := \bar{B}^i \cap Y_{\bar{D}^i}, \quad F^i := \bar{B}^i \cap Y_{u_i}.$$

With appropriate deformation retractions and excisions, one can prove the following homology direct sum decomposition formula which is also true in more general setting (cf. [20, 21, 24, 10]). In this paper we consider homology with $\mathbb{Z}$-coefficient.

2.2. **Proposition.** (Additivity of vanishing homology) *With the notations and assumptions as above, we have*

$$H_*(E, \hat{F}) \cong \bigoplus_{i=0}^{\sigma} H_*(E^i, F^i). \qquad \diamond$$

2.3. **Decomposition of the fibre in homotopy.** We make a homotopy model of the wedge of all the local Milnor fibres F^i. Denote

$$\Gamma = \bigcup_{i=1}^{\sigma} c_i, \quad D' = \bigcup_{i=0}^{\sigma} \bar{D}^i, \quad E^* = Y_{D' \cup \Gamma} \overset{h}{\simeq} Y_D, \quad F^* = Y_\Gamma \overset{h}{\simeq} Y_{u_0} = \hat{F},$$

where and in the following, $\overset{h}{\simeq}$ means "is homotopy equivalent to".

In the fibre F^* one sees the following:

1) $F^0, \ldots, F^\varsigma$, the local Milnor fibres of f at $P_0, \ldots, P_\varsigma$;

2) the vanishing cycles from each $F^{\varsigma+j} \overset{h}{\simeq} S^n \vee \cdots \vee S^n$ ($\beta_{\varsigma+j}$ copies of n-sphere), the local Milnor fibre of f at each $P_{\varsigma+j} \in Y_{\mathrm{reg}}$, where $\beta_{\varsigma+j}$ is the local Milnor number of f at $P_{\varsigma+j}$.

Let $h_1^{\varsigma+j} \cup \cdots \cup h_{\beta_{\varsigma+j}}^{\varsigma+j}$ be the $(n+1)$-cells (called the thimbles) to be attached to $F^{\varsigma+j}$ in order to kill the vanishing cycles. Let H be the union of all the thimbles over all j.

Assume that $\hat{F}$ and F^i $(0 \leq i \leq \varsigma)$ are connected. Let $x_i \in \partial F^i$ $(0 \leq i \leq \varsigma)$, and let $x_{\varsigma+j} \in F^{\varsigma+j}$ be the wedge point of the spheres. Take a non self-intersecting path γ_i in F^* connecting x_0 and x_i $(1 \leq i \leq \sigma)$ by lifting c_i (if $b_i \neq b_0$) or within $Y_{u_0} = \hat{F}$ (if $b_i = b_0$), so that two paths intersect only at x_0. We also want that γ_i does not intersect F^j, for $j \neq i$. In order to satisfy this condition, we may need to modify the path within a tubular neighbourhood of F^*, resp. $\hat{F}$, which is of course possible. We then have the inclusion

$$\iota : F' := F^0 \cup (\gamma_1 \cup F^1) \cup \cdots \cup (\gamma_\sigma \cup F^\sigma) \hookrightarrow F^*.$$

Note that $F' \overset{h}{\simeq} F^+ \vee S$, where

$$F^+ := F^0 \cup (\gamma_1 \cup F^1) \cup \cdots \cup (\gamma_\varsigma \cup F^\varsigma) \overset{h}{\simeq} F^0 \vee F^1 \vee \cdots \vee F^\varsigma,$$

and $S := S_1^n \vee \cdots \vee S_\beta^n$ is the wedge of $\beta = \sum_j \beta_{\varsigma+j}$ copies of the n-sphere.

Let B'_j be the ball with boundary S^n_j in the bouquet of spheres. Then we have the inclusion

$$F' \hookrightarrow F^+ \vee B' := F^+ \vee B'_1 \vee \cdots \vee B'_\beta.$$

Define $\varphi : F^+ \vee S \hookrightarrow F^*$ by the composition of $F^+ \vee S \overset{h}{\simeq} F' \overset{\iota}{\hookrightarrow} F^*$. From the identification of balls with thimbles we obtain the following maps

$$\varphi' : F^+ \vee B' \longrightarrow F^* \cup H, \quad \varphi'' : F^+ \hookrightarrow F^+ \vee B' \longrightarrow F^* \cup H.$$

2.4. **Theorem.** *Under the above assumptions, if E is contractible, then the map φ induces isomorphisms on all the homology groups*

$$H_*(F^+ \vee S) \cong H_*(F^*).$$

Moreover, if $F^0, \dots, F^\varsigma$ and $\hat{F}$ are simply connected, then

$$\hat{F} \overset{h}{\simeq} F^+ \vee S \overset{h}{\simeq} F^0 \vee \cdots \vee F^\varsigma \vee S.$$

Proof We follow the proof of [21, Proposition 2.8]. The maps φ and φ' above give a map between the space pairs:

$$\varphi^{rel} : (F^+ \vee B', F^+ \vee S) \longrightarrow (F^* \cup H, F^*),$$

which induces the map between the homology groups:

$$\begin{array}{ccccc}
H_{q+1}(F^+ \vee B', F^+ \vee S) & \longrightarrow & H_q(F^+ \vee S) & \longrightarrow & H_q(F^+ \vee B') \\
\downarrow \varphi_*^{rel} & & \downarrow \varphi_* & & \downarrow \varphi'_* \\
H_{q+1}(F^* \cup H, F^*) & \longrightarrow & H_q(F^*) & \longrightarrow & H_q(F^* \cup H).
\end{array}$$

Diagram 1

By excision, φ_*^{rel} is an isomorphism (cf. [11, §3]). Note that $F^+ \overset{h}{\simeq} F^+ \vee B'$. By mainly excisions, it follows that the inclusion $(E^+, F^+) \hookrightarrow (E^*, F^* \cup H)$ induces an isomorphism in homology, where

$$E^+ := \left(\bar{B}^0 \cap Y \cap f^{-1}(\bar{D}^0)\right) \cup \left(\bigcup_{i=1}^{\varsigma} \gamma_i\right) \cup \left(\bigcup_{i=1}^{\varsigma} \bar{B}^i \cap Y \cap f^{-1}(\bar{D}^i)\right).$$

Hence φ'_* is an isomorphism since both E^+ and E^* are contractible. These imply that φ_* is an isomorphism. ◇

We return to our original settings. Let $(X, 0)$ be an analytic space germ of dimension $n+1 > 2$, locally embedded in $(\mathbb{C}^{m+1}, 0)$. Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ be a function germ. Let l be a linear function, considered as the last coordinate of $\mathbb{C}^{m+1}$, and denote $X_t = X \cap l^{-1}(t)$. The definition (Definition 1.1) of one-parameter deformation (X_t, f_t) of an isolated singularity (X_0, f_0) implies the following facts:

(1) $l^{-1}(0)$ intersects all the strata of $X \setminus \{0\}$ transversally. Note that the strata of dimensions less than 2 are contained in $\Sigma_{\mathcal{S}}(f,l)$. For any stratum $S_i \in \mathcal{S}$ of dimension at least 2 and any point $z \in S_i \cap l^{-1}(0)$, if the transversality fails, then z is a critical point of $l|_{S_i}$, the restriction of l to S_i. Since the critical locus $\Sigma l|_{S_i}$ of $l|_{S_i}$ is contained in $\Sigma(f,l)|_{S_i} \subset \Sigma_{\mathcal{S}}(f,l)$ and $l^{-1}(0) \cap \Sigma_{\mathcal{S}}(f,l) = \{0\}$, we have $z = 0$;
(2) $l^{-1}(0)$ is transversal to all the strata of $f^{-1}(0) \setminus \{0\}$.

By applying Proposition 2.2 to (X_t, f_t), we see immediately from the following Lemma 2.5 that Conjecture A is true in homology.

2.5. **Lemma.** *Let (X_t, f_t) be a one-parameter deformation of the isolated singularity (X_0, f_0). Then we have*

1) *For any $\varepsilon > 0$ small enough, $f_0^{-1}(0) \cap X_0$ is transversal to the boundary $\partial \bar{B}_\varepsilon$ of the closed ball $\bar{B}_\varepsilon \subset \mathbb{C}^m$ with center 0 and radius ε. We denote this by $(f_0^{-1}(0) \cap X_0) \bar{\pitchfork} \partial \bar{B}_\varepsilon$;*
2) *Fix an ε_0 with the property in 1). There exist $\eta > 0, \tau > 0$ such that for any $|u| < \eta$ and $|t| < \tau$, we have $(f_t^{-1}(u) \cap X_t) \bar{\pitchfork} \partial \bar{B}_0$, where $\bar{B}_0 := \bar{B}_{\varepsilon_0}$;*
3) *Let $\varepsilon_0 > 0$, $\eta_0 > 0$ and $\tau_0 > 0$ be as in 2). If τ_0 is small enough, then for any $|t| < \tau_0$ and $u \in \partial \bar{D}_0$, $\hat{F} := f_t^{-1}(u) \cap X_t \cap \bar{B}_0 \overset{h}{\simeq} F_0 := f_0^{-1}(u) \cap X_0 \cap \bar{B}_0$, where $\bar{D}_0$ is the closure of the open disk in $\mathbb{C}$ with center 0 and radius η_0;*
4) *$f_t^{-1}(\bar{D}_0) \cap X_t \cap \bar{B}_0 \overset{h}{\simeq} f_0^{-1}(\bar{D}_0) \cap X_0 \cap \bar{B}_0$. In particular, if η_0 is small enough, both spaces are contractible.*

Proof Part 1) is the well known lemma of the "conic structure of the analytic germs", see, for instance, [19] for the smooth case and [3] for the stratified case.

The proof of statement 2) follows from [12, §2]. Note that the conditions required for l in loc. cit. such that the proof works are fulfilled by our l, since $\dim \Sigma_{\mathcal{S}}(f,l) \leq 1$ (see also the similar remarks in [24, §1.1]).

To prove 3), we consider the map (cf. [12, 13])

$$G =: (f,l) : X \cap (\bar{B}_0 \times \Delta) \longrightarrow \mathbb{C} \times \Delta,$$

where Δ is the open disc in $\mathbb{C}$ with center 0 and radius τ_0. Let

$$Z_i^{(1)} = G^{-1}(\partial \bar{D}_0 \times \Delta) \cap \mathcal{S}_i \cap (B_0 \times \Delta), \quad \text{and} \quad Z_i^{(2)} = G^{-1}(\partial \bar{D}_0 \times \Delta) \cap \mathcal{S}_i \cap (\partial \bar{B}_0 \times \Delta)$$

be the strata of the Whitney stratification of $G^{-1}(\partial \bar{D}_0 \times \Delta) \cap X \cap (\bar{B}_0 \times \Delta)$ induced from $\mathcal{S} = \{\mathcal{S}_i\}_i$. Obviously, each $G|Z_i^{(1)}$ is a submersion. By the transversality 2), each $G|Z_i^{(2)}$ is again a submersion. It follows from Thom-Mather's first isotopy lemma that 3) holds.

Let $Z := G^{-1}(\bar{D}_0 \times \Delta) \cap X \cap (\bar{B}_0 \times \Delta) \longrightarrow \Delta$ be the map $\pi \circ G$, where $\pi : \bar{D}_0 \times \Delta \longrightarrow \Delta$ is the projection to the second component. Stratify Z by $Z_i^{(1)}, Z_i^{(2)}$ and

$$Z_i^{(3)} = G^{-1}(D_0 \times \Delta) \cap \mathcal{S}_i \cap (B_0 \times \Delta), \quad Z_i^{(4)} = G^{-1}(D_0 \times \Delta) \cap \mathcal{S}_i \cap (\partial \bar{B}_0 \times \Delta).$$

It is clear that $\{Z_i^{(j)}\}$ is a Whitney stratification of Z and the restrictions of π to each stratum is a submersion. By Thom-Mather's first isotopy lemma, π is a locally trivial topological fibration. This proves 4). $\diamond$

In some cases, one can remove from Theorem 2.4 the requirement that the Milnor fibres F^i be simply connected, and get a bouquet decomposition in homotopy. For example, if one can prove that the map φ induces isomorphisms on the fundamental groups of the spaces and on the homologies of the universal coverings of the spaces, then use Whitehead's theorem [28]. This is the approach of Siersma [21]. In the remainder of this section we use this idea to prove that under some assumptions Conjecture A is also true in homotopy.

Define $\rho : \mathbb{C}^{m+1} \longrightarrow \mathbb{R}$ by $\rho(z_1, \dots, z_m, z_{m+1}) := \sum_{j=1}^{m} z_j \bar{z}_j$. Still denote by ρ its restriction to X. Denote by $\Gamma_{\mathcal{S}}(\rho, l)$ the germ of the set $\overline{\Sigma_{\mathcal{S}}(\rho, l) \setminus \rho^{-1}(0)}$ at the origin, where $\Sigma_{\mathcal{S}}(\rho, l)$ denotes the critical set of the map $(\rho, l) : X \longrightarrow \mathbb{R} \times \mathbb{C}$ relative to the stratification $\mathcal{S}$ of X.

2.6. **Definition.** Let X_t be a space family with $0 \in X_t$ for all $t \in \mathbb{C}$. Identify $\mathrm{Cone}\,(X_t \cap \partial B_0)$ with $(X_t \cap \partial B_0) \times [0, \varepsilon_0]/(x, 0) \sim (y, 0)$, where B_0 is the open ball with center 0 and radius ε_0. If there exist $\varepsilon_0 > 0$ and $\tau_0 > 0$ such that for each $|t| < \tau_0$, there exists a homeomorphism $\varkappa_t$ from $\mathrm{Cone}\,(X_t \cap \partial \bar{B}_0)$ to $X_t \cap \bar{B}_0$ such that $\rho \circ \varkappa_t$ is the projection onto the interval $[0, \varepsilon_0]$, we say that the family of germs $(X_t, 0)$ has *link stability.*

2.7. **Lemma.** *If $\Gamma_{\mathcal{S}}(\rho, l) = \emptyset$, then the family $(X_t, 0)$ has link stability.*

Proof Let W be an open neighborhood of the origin of $\mathbb{C}^{m+1}$ such that inside $W \cap X$, $\Gamma_{\mathcal{S}}(\rho, l) = \emptyset$. There exist $\varepsilon_0 > 0, \tau_0 > 0$ such that $\bar{B}_0 \times \Delta \subset W$, where B_0 is the open ball in $\mathbb{C}^m$ with center 0 and radius ε_0, and Δ is the open disc in $\mathbb{C}$ with center 0 and radius τ_0. Since $\Gamma_{\mathcal{S}}(\rho, l) = \emptyset$, for each $t \in \Delta$, the restriction of ρ to each stratum of X_t is a submersion, except at the origin. In other words, there are no 0-dimensional strata of X_t except the origin $(0, t)$ and each positive dimensional stratum of X_t intersects $\partial \bar{B}_\varepsilon$ transversally, for each $0 < \varepsilon \leq \varepsilon_0$.

By [6, II (3.3)] or [7, p. 42], there exists a controlled vector field v on a punctured neighborhood $U \setminus \{0\}$ of $\bar{B}_0 \setminus \{0\}$, tangent to the strata of X_t, such that

$$d_z\rho(v) = -\left(\frac{d}{ds}\right)_\rho,$$

where U is an open neighborhood of $\bar{B}_0$ and $\left(\frac{d}{ds}\right)_\rho$ is the unit tangent vector to $\mathbb{R}$ at ρ.

By [6, II (4.7)] or [7, p. 42], this vector field v can be integrated, and by choosing the initial values appropriately, we can get the desired homeomorphism.

More precisely, let $y \in X_t \cap \partial \bar{B}_0$, and $h_y : (-\delta, \delta) \longrightarrow X_t \setminus \{0\}$ be the integral curve of v with $h_y(0) = y$. The following points are important for the integration of the controlled vector fields. For each $s \in (-\delta, \delta)$, $h_y(s)$ is the unique point on the orbit passing through y, and $h_y(-\delta, \delta)$ is in the same stratum which contains y.

On each stratum, h_y is smooth, and $\frac{d(\rho \circ h_y)}{ds}(s) = -1$, so $\rho \circ h_y(s) = \rho(y) - s = \varepsilon_0 - s$. Define

$$\varkappa_t : (0, \varepsilon_0] \times (X_t \cap \partial \bar{B}_0) \longrightarrow (X_t \cap \bar{B}_0) \setminus \{0\}, \quad \text{by} \quad (s, y) \longmapsto h_y(\varepsilon_0 - s).$$

Then $\rho \circ \varkappa_t(s, y) = s$. This also shows that, $\varkappa_t$ is smooth on each stratum, and $\rho \circ \varkappa_t$ is the projection onto $(0, \varepsilon_0]$. Hence $\varkappa_t$ can be extended to a homeomorphism between $\mathrm{Cone}(X_t \cap \partial \bar{B}_0)$ and $X_t \cap \bar{B}_0$. $\diamond$

2.8. **Remark.** The assumption $\Gamma_{\mathcal{S}}(\rho, l) = \emptyset$ in Lemma 2.7 is satisfied automatically in some cases: $X = X_0 \times \mathbb{C}$, or X_t is a family of weighted homogeneous complete intersections with isolated singularity. If X_t is a family of hypersurfaces with isolated singularities and does not split, then this assumption implies the topological triviality of the family. This is similar to the so-called (m) condition for the pair $(X \setminus (0 \times \mathbb{C}), 0 \times \mathbb{C})$ used in the literature (see, *e.g.*, [1]). However, this condition does not imply that the pair $(X \setminus (0 \times \mathbb{C}), 0 \times \mathbb{C})$ satisfies Whitney condition as shown by the following example.

2.9. **Example.** Let X be the Briançon-Speder family of surfaces defined by $h = z^3 + ty^{2\alpha+1}z + xy^{3\alpha+1} + x^{6\alpha+3} = 0$ $(\alpha \geq 1)$. Then $(X \setminus (0 \times \mathbb{C}), 0 \times \mathbb{C})$ does not satisfy the Whitney condition (cf. [2]). Computation shows that $\Gamma_{\mathcal{S}}(\rho, l) = \emptyset$ (see also Example 3.5).

2.10. **Theorem.** *Let (X_t, f_t) be a one-parameter deformation of the isolated singularity (X_0, f_0), with X_t irreducible at 0 and $\dim X_t = n+1 \neq 3$, $\forall t$, $n \geq 1$. Suppose that there exist $\varepsilon_0 > 0$ and $\tau_0 > 0$ such that $X_t \cap B_0 \setminus \{0\}$ is non-singular for all $t \in \Delta$, and that $l^{-1}(t) \cap \Sigma_{\mathcal{S}}(f, l) := \{P_0(t) = 0, P_1(t), \dots, P_\sigma(t)\}$ for $t \in \Delta \setminus \{0\}$ and $\lim_{t\to 0} P_j(t) = 0$. If $(X_t, 0)$ has link stability, then*

$$F_0 \overset{h}{\simeq} F_t \vee S^n \vee \cdots \vee S^n,$$

where F_t is the Milnor fibre of f_t at 0, and the total number of spheres S^n in the bouquet is equal to the sum of the local Milnor numbers of f_t at $P_i(t)$.

Proof We use Theorem 2.4 and Lemma 2.5. Let $\varepsilon_t > 0$ and $\eta_t > 0$ be the Milnor data for f_t, i.e., the restriction $f_t : \bar{B}_t \cap X_t \cap f_t^{-1}(\bar{D}_t^*) \longrightarrow \bar{D}_t^* := \bar{D}_t \setminus 0$ of f_t is the Milnor fibration of f_t, where B_t is an open ball with center 0 and radius ε_t, and D_t is an open disc with center 0 and radius η_t. We also assume that $0 < \varepsilon_t < \varepsilon_0$ and $0 < \eta_t < \eta_0$ for $t \neq 0$.

We briefly recall the constructions in §2.1 and §2.3 for (X_t, f_t). By the assumptions, there exists $\tau_0 > 0$ such that for any $t \in \Delta$, X_t has an isolated singularity in $\mathcal{B} := X_t \cap \bar{B}_0$.

Set $b_0 := f_t(P_0(t)) = 0$, $b_i := f_t(P_i(t)) \in D_0$. Note that $\varsigma = 0$, since X_t has an isolated singularity at $P_0(t) = 0$ in $\mathcal{B}$. In D_0, take small closed discs $\bar{D}^i$ with center b_i and radius $\eta' > 0$. Let u_i be a point on $\partial\bar{D}^i$. For $i > 0$, let c_i be the path connecting u_0 with u_i, as explained in §2.3.

Let $\bar{B}^i$ be the closed ball with center $P_i(t)$ and radius $\varepsilon' > 0$. Take $\eta' > 0, \varepsilon' > 0$ so small that $\bar{B}^i$ (resp. $\bar{D}^i$) are disjoint and contained in B_0 (resp. D_0), the restriction of f_t to $X_t \cap \bar{B}^i \cap f_t^{-1}(\bar{D}^i \setminus \{b_i\})$ is a Milnor fibration with fibre $F^i := X_t \cap \bar{B}^i \cap f_t^{-1}(u_i)$, and $E^i := \mathcal{B} \cap \bar{B}^i \cap f^{-1}(\bar{D}^i)$ is contractible.

Similarly, one has E, $\hat{F}$, E^*, $F^* \overset{h}{\simeq} \hat{F}$, $F^+ = F^0$, and the mapping $\varphi : F^0 \vee S \hookrightarrow F^*$. Note that F^i is connected, since X_t is irreducible at 0 and with isolated singularity, $\forall t$.

Since $X_t \cap B_0$ has an isolated singularity at 0, $F^+ = F^0$ is the Milnor fibre F_t of f_t. If $\dim X_t = 2$, by using resolution of singularities, one can prove that F_t is a bouquet of one-spheres. Hence, the theorem follows. If $\dim X_t > 3$ and F_t is simply connected, then the theorem also follows from Theorem 2.4 and Lemma 2.5.

In the general case, we make use of Whitehead's theorem [28] in a similar manner as done by Siersma in [21]. Namely, we prove the following two statements in the remainder of this section:

1) the map φ induces isomorphism on the fundamental groups (cf. Lemma 2.11);
2) the map φ induces isomorphism on the homology groups of universal coverings of the spaces (cf. Proposition 2.16).

Then we may apply Whitehead's theorem [28] to conclude that φ is a homotopy equivalence. $\diamond$

2.11. **Lemma.** *Let (X_t, f_t) be a one-parameter deformation of the isolated singularity (X_0, f_0) with $\dim X_t > 3$. If $X_t \cap B_0$ has an isolated singularity at 0 and $(X_t, 0)$ has link stability, then*

1) $\pi_1(\partial F_t) \cong \pi_1(\partial F^*)$, *where* $\partial F^* := F^* \cap \partial \bar{B}_0$;
2) $\pi_1(F_t) \cong \pi_1(F^*)$, *hence* $\pi_1(F_t \vee S) \cong \pi_1(F^*)$.

Proof Since $\hat{F}$ is a deformation retract of F^* (cf. §2.3), it is enough to prove the lemma by replacing F^* by $\hat{F}$; i.e.

1′) $\pi_1(\partial F_t) \cong \pi_1(\partial \hat{F})$;
2′) $\pi_1(F_t) \cong \pi_1(\hat{F})$, hence $\pi_1(F_t \vee S) \cong \pi_1(\hat{F})$.

We follow [21, §3] closely. In the proof, we use the following notations:

$$\begin{array}{lll} M = \partial \bar{B}_0 \cap X_t, & \hat{F} = f_t^{-1}(u_0) \cap \bar{B}_0 \cap X_t, & K = f_t^{-1}(0) \cap \partial \bar{B}_0 \cap X_t, \\ M_t = \partial \bar{B}_t \cap X_t, & F_t = f_t^{-1}(u_0) \cap \bar{B}_t \cap X_t, & K_t = f_t^{-1}(0) \cap \partial \bar{B}_t \cap X_t, \\ \nabla F = \overline{\hat{F} \setminus F_t}, & \nabla M = \overline{(B_0 \setminus B_t) \cap X_t}. & \end{array}$$

Note that we have $K \overset{h}{\simeq} \partial \hat{F}$ and $K_t \overset{h}{\simeq} \partial F_t$.

$$\begin{tikzcd} \pi_1(M_t) \arrow[r, "\psi_1"] & \pi_1(\nabla M) & \pi_1(M) \arrow[l, "\psi_2"'] \\ \pi_1(K_t) \arrow[u, "\cong"] \arrow[r, "\phi_2"] \arrow[d, "\cong"] & \pi_1(\nabla F) \arrow[u, "\phi_1"] \arrow[d, "\cong"] & \pi_1(K) \arrow[u, "\cong"] \arrow[l, "\cong"'] \\ \pi_1(F_t) \arrow[r, "\phi_3"] & \pi_1(\hat{F}) & \end{tikzcd}$$

Diagram 2

All the morphisms in Diagram 2 are induced by the inclusion maps. The indicated isomorphisms can be proved via Morse theory, by using the results of Hamm [8, 2.9]. By link stability, the inclusions of M_t and M into ∇M are homotopy equivalences. We have the isomorphisms ψ_1 and ψ_2. It follows that ϕ_1 and ϕ_2 are isomorphisms. Hence ϕ_3 is an isomorphism. $\diamond$

2.12. Let (X_t, f_t) be a one-parameter deformation of the isolated singularity (X_0, f_0) with $\dim X_t > 3$. Assume $X_t \cap B_0$ has an isolated singularity at 0 and $(X_t, 0)$ has link stability. We continue to use the notations in §2.10 and §2.11.

By link stability, the cone cM over M is homeomorphic to $\mathcal{B}$. Let $\tilde{M}$ be the universal covering of M. $\tilde{M}$ is smooth, connected, and simply connected. Set $\tilde{\mathcal{B}} := c\tilde{M}$, the cone over $\tilde{M}$, which is smooth outside the top $*$. There is a map $\pi : \tilde{\mathcal{B}} \longrightarrow \mathcal{B}$ compatible

with the cone structure such that the restriction of π to $\tilde{\mathcal{B}} \setminus *$ is also a covering, which can be identified with $(0,1] \times \tilde{M} \longrightarrow (0,1] \times M$. The function f_t on $\bar{B}_0 \cap X_t$ and its restriction to M can be lifted to functions on $\tilde{\mathcal{B}}$ and $\tilde{M}$ respectively; i.e., we have commutative Diagram 3.

$$\begin{tikzcd}
\tilde{M} \arrow[r, hook] \arrow[d, "\pi"] & \tilde{\mathcal{B}} \arrow[d, "\pi"] \arrow[dr, "\tilde{f}_t := f_t \circ \pi"] & \\
M \arrow[r, hook] & \mathcal{B} \arrow[r, "f_t"] & \mathbb{C}
\end{tikzcd}$$

Diagram 3

2.13. **Lemma.** *Under the assumptions above, $\tilde{K} = \tilde{f}_t^{-1}(0) \cap \tilde{M}$ and $\tilde{\hat{F}} = \tilde{f}_t^{-1}(u_0) \cap \tilde{\mathcal{B}}$ are simply connected. Moreover, the restrictions of π give universal coverings*

$$\pi : \tilde{K} \longrightarrow K \quad \text{and} \quad \pi : \tilde{\hat{F}} \longrightarrow \hat{F}.$$

Proof One uses the following fact from topology (see [21, Lemma 4.1]).
Sublemma Let Y be connected, locally path connected and locally simply connected. Let $\pi : \tilde{Y} \longrightarrow Y$ be the universal covering of Y. If $Z(\subset Y)$ is connected and the inclusion map $Z \hookrightarrow Y$ induces an isomorphism on the fundamental groups, then the restriction of π to $\tilde{Z} := \pi^{-1}(Z)$ is also a universal covering.

The space $\mathcal{B} \setminus \{0\} \overset{h}{\simeq} M$ satisfies the requirements for Y in the sublemma. By Lemma 2.11, both K and $\hat{F}$ satisfy the requirements for Z in the sublemma, and the lemma follows. ◇

2.14. We repeat the constructions in §2.1 and §2.3 for the spaces $\tilde{\mathcal{B}}, \tilde{M}, \tilde{\hat{F}}$ and $\tilde{K}$ for the function $\tilde{f}_t$. At the same time, we also use the notations in §2.10 and §2.11.

Note that, by Lemma 2.5, for $\varepsilon_0 > 0, \eta_0 > 0$ and Δ small, for each $t \in \Delta$, $f_t : \mathcal{B} \cap f_t^{-1}(\bar{D}_0 \setminus \{b_0, \dots, b_\sigma\}) \longrightarrow \bar{D}_0 \setminus \{b_0, \dots, b_\sigma\}$ is a locally trivial topological fibration. Hence

$$\tilde{f}_t : \tilde{\mathcal{B}} \cap \tilde{f}_t^{-1}(\bar{D}_0 \setminus \{b_0, \dots, b_\sigma\}) \longrightarrow \bar{D}_0 \setminus \{b_0, \dots, b_\sigma\}$$

is also a locally trivial topological fibration. Denote $\tilde{E} = \tilde{\mathcal{B}} \cap \tilde{f}_t^{-1}(\bar{D}_0)$, $\tilde{E}^i = \pi^{-1}(E^i)$, $\tilde{F}^i = \pi^{-1}(F^i)$ and $\tilde{F}^* = \pi^{-1}(F^*)$. For $i > 0$, $\tilde{E}^i$ is a disjoint union of closed sets, and each of which is homeomorphic to E^i. And $\tilde{F}^i$ is a disjoint union of closed sets, each of which is homeomorphic to $F^i = X \cap \bar{B}^i \cap f_t^{-1}(u_i)$. These are possible since the restriction of π to $\tilde{\mathcal{B}} \setminus *$ is a universal covering.

Then the following proposition is similar to Proposition 2.2 and can be proved in the same way.

2.15. **Proposition.** *With the notations and assumptions above, we have*

$$H_*(\tilde{E}, \tilde{\hat{F}}) \cong \bigoplus_{i=0}^{\sigma} H_*(\tilde{E}^i, \tilde{F}^i). \qquad ◇$$

Note that the ball B^0 is a Milnor ball of f_t at $P_0 = 0$. We have $F^0 = F_t$, the Milnor fibre of f_t. The lifting $\tilde{F}^0 = \tilde{f}_t^{-1}(u_0) \cap \tilde{B}^0$ of F^0 by π is a universal covering of F^0 by the sublemma, where $\tilde{B}^0$ is the lifting of $\bar{B}^0 \cap X_t$ by π.

The fibre $\tilde{F}^*$ contains $\tilde{F}^0$ and $\tilde{F}^i$ ($i > 0$). For each $i > 0$, $\tilde{F}^i$ is a disjoint union of pieces which are copies of F^i. And each F^i is homotopy equivalent to a bouquet of spheres $F^i \overset{h}{\simeq} S^n \vee \cdots \vee S^n$ (β_i copies). Let $h_1^i \cup \cdots \cup h_{\beta_i}^i$ be the union of the thimbles. Denote $H = \bigcup_{j=1}^{\sigma} (h_1^j \cup \cdots \cup h_{\beta_j}^j)$. Let x_i be the wedge point in F^i, and $x_0 \in \partial \tilde{F}^0$. Let $\tilde{\gamma}_i$ be the union of the paths in $\tilde{F}^*$ connecting x_0 and the liftings of x_i in a usual way. One can take the liftings of c_i as $\tilde{\gamma}_i$. We have the inclusion:

$$\tilde{F}' := \tilde{F}^0 \cup (\tilde{\gamma}_1 \cup \tilde{F}^1) \cup \cdots \cup (\tilde{\gamma}_\sigma \cup \tilde{F}^\sigma) \hookrightarrow \tilde{F}^*.$$

Obviously $\tilde{F}'$ is homotopy equivalent to $\tilde{F}^0 \vee \tilde{S}$, where $\tilde{S}$ is a wedge of the lifted bouquets in F^i's ($i > 0$).

Denote by $\tilde{H} = \bigsqcup H$ the disjoint union of H such that the attachments of the balls in $\tilde{H}$ to the spheres in $\tilde{S}$ will kill all the the n-spheres in $\tilde{F}^*$ coming from the liftings of F^i ($i > 0$). The result of this attachment is denoted by $\tilde{F}^* \cup \tilde{H}$.

Let B_j' be the ball with boundary S_j^n, then we have the inclusion

$$F^0 \hookrightarrow F^0 \vee B' := F^0 \vee B_1' \vee \cdots \vee B_\beta', \quad \text{with } \beta = \sum_{i=1}^{\sigma} \beta_i.$$

Denote by $\tilde{B}'$ the disjoint union $\bigsqcup (B_1' \vee \cdots \vee B_\beta')$. Using the union of the paths $\tilde{\gamma}_i$ above, we have $\tilde{\varphi} : \tilde{F}^0 \vee \tilde{S} \hookrightarrow \tilde{F}^*$, the composition of $\tilde{F}^0 \vee \tilde{S} \overset{h}{\simeq} \tilde{F}' \hookrightarrow \tilde{F}^*$, and $\tilde{F}^0 \hookrightarrow \tilde{F}^0 \vee \tilde{B}'$. We also have the following obvious mappings

$$\tilde{\varphi}' : \tilde{F}^0 \vee \tilde{B}' \longrightarrow \tilde{F}^* \cup \tilde{H}, \quad \tilde{\varphi}'' : \tilde{F}^0 \hookrightarrow \tilde{F}^0 \vee \tilde{B}' \longrightarrow \tilde{F}^* \cup \tilde{H}.$$

With the data above, we have the following conclusion similar to Theorem 2.4 and its proof is almost word by word the same as that of Theorem 2.4.

2.16. **Proposition.** *The map*

$$\tilde{\varphi} : \tilde{F}^0 \vee \tilde{S} \longrightarrow \tilde{F}^*$$

induces isomorphisms on all the homology groups:

$$H_*(\tilde{F}^0 \vee \tilde{S}) \cong H_*(\tilde{F}^*). \qquad \diamond$$

3. The Lê-Ramanujam problem

3.1. Let $\varepsilon_t > 0, \eta_t > 0$ be admissible for the Milnor fibration of the germ of f_t at 0. Denote by B_t the open ball in $\mathbb{C}^m$ with center 0 and radius ε_t, and by D_t the open disc in $\mathbb{C}$ with center 0 and radius η_t. Denote by Δ a small open disc in $\mathbb{C}$ with center 0.

3.2. **Theorem.** *Let (X_t, f_t) be a one-parameter deformation of the isolated singularity (X_0, f_0) with X_t irreducible at 0 and $\dim X_t \neq 3$. Suppose there exists an open neighborhood U of 0 such that, for each $t \in \Delta$, $U \cap X_t \setminus \{0\}$ is non-singular. Suppose further that $(X_t, 0)$ has link stability and the isolated singularity (X_0, f_0) does not split. Then*

1) *the homotopy type of the Milnor fibre of f_t is constant; i.e. for any $u \in \bar{D}_t^* := \bar{D}_t \setminus \{0\}$*
$$F_0 = f_0^{-1}(u) \cap X_0 \cap \bar{B}_0 \overset{h}{\simeq} F_t = f_t^{-1}(u) \cap X_t \cap \bar{B}_t;$$
2) *The monodromy fibrations of f_0 and f_t are fibre homotopy equivalent (as fibrations over $\partial\bar{D}_0$ and $\partial\bar{D}_t$ respectively); i.e.*
$$E_0 := f_0^{-1}(\partial\bar{D}_0) \cap X_0 \cap \bar{B}_0 \quad \overset{h}{\simeq} \quad E_t := f_t^{-1}(\partial\bar{D}_t) \cap X_t \cap \bar{B}_t.$$
Here and in the following, we denote the fibration $(E_t, f_t|E_t, \partial\bar{D}_t)$ by E_t;

If, moreover, the Milnor fibre F_t of f_t is simply connected or if $\dim X_t = 2$, *then we have:*

3) *the diffeomorphism type of the Milnor fibration of f_t is constant (as fibrations over $\partial\bar{D}_0$ and $\partial\bar{D}_t$ respectively); i.e.*
$$E_0 := f_0^{-1}(\partial\bar{D}_0) \cap X_0 \cap \bar{B}_0 \quad \overset{\text{diffeo}}{\simeq} \quad E_t := f_t^{-1}(\partial\bar{D}_t) \cap X_t \cap \bar{B}_t;$$
4) *the local topological type of f_t is constant; i.e.,*
$$\left(\bar{B}_0 \cap X_0, \bar{B}_0 \cap X_0 \cap f_0^{-1}(0)\right) \overset{homeo}{\simeq} \left(\bar{B}_t \cap X_t, \bar{B}_t \cap X_t \cap f_t^{-1}(0)\right).$$

Proof We follow the pattern of Lê-Ramanujam's proof [14]. By the non-splitting condition, f_t has no critical point on $U \cap X_t \setminus \{0\}$ for any t.

Note that the diffeomorphism type of the Milnor fibration does not depend on the choice of $\eta > 0$. Hence for any $t \neq 0$ fixed, as fibrations over $\partial\bar{D}_0$ and $\partial\bar{D}_t$ respectively, we have a diffeomorphism
$$E_0 = f_0^{-1}(\partial\bar{D}_0) \cap X_0 \cap \bar{B}_0 \quad \overset{\text{diffeo}}{\simeq} \quad E_0' := f_0^{-1}(\partial\bar{D}_t) \cap X_0 \cap \bar{B}_0.$$
Then we prove that (as fibrations)
$$E_0' = f_0^{-1}(\partial\bar{D}_t) \cap X_0 \cap \bar{B}_0 \quad \overset{\text{diffeo}}{\simeq} \quad E_t' := f_t^{-1}(\partial\bar{D}_t) \cap X_t \cap \bar{B}_0$$
So there is an inclusion $E_t \hookrightarrow E_t'$, and also inclusions for their fibres. We prove these induce the desired results.

Consider the map G defined in the proof of Lemma 2.5
$$G(z,t) = (f(z,t), t) : X \cap (\bar{B}_0 \times \Delta) \longrightarrow \mathbb{C} \times \Delta.$$
This map induces the following two differentiable fibrations by Ehresmann's theorem (see [11]):
$$G_1 : \left(\bar{B}_0 \times \Delta\right) \cap X \cap G^{-1}\left(\partial\bar{D}_t \times \Delta\right) \longrightarrow \partial\bar{D}_t \times \Delta,$$
and
$$G_2 : \left(\partial\bar{B}_0 \times \Delta\right) \cap X \cap G^{-1}\left(\bar{D}_t \times \Delta\right) \longrightarrow \bar{D}_t \times \Delta.$$
Moreover, G_2 is a trivial fibration since $\bar{D}_t \times \Delta$ is contractible.

Hence, as fibrations over $\partial\bar{D}_t \times 0$ and $\partial\bar{D}_t \times t$ respectively,
$$G^{-1}(\partial\bar{D}_t \times 0) \quad \overset{\text{diffeo}}{\simeq} \quad G^{-1}(\partial\bar{D}_t \times t),$$

and this is compatible with the trivialization G_2. This proves that as fibrations over $\partial \bar{D}_t$

$$E'_0 = f_0^{-1}(\partial \bar{D}_t) \cap X_0 \cap \bar{B}_0 \overset{\text{diffeo}}{\simeq} E'_t := f_t^{-1}(\partial \bar{D}_t) \cap X_t \cap \bar{B}_0.$$

Next, we prove that $E'_t \overset{\text{diffeo}}{\simeq} E_t$ as fibrations over $\partial \bar{D}_t$. Recall that a fibre of E_t is denoted by F_t, which is the Milnor fibre of f_t. Since the fibre of E'_t is diffeomorphic to the Milnor fibre F_0 of f_0, in the following we use this notation.

Obviously $E_t \hookrightarrow E'_t$. By Ehresmann's theorem (loc. cit.)

$$f_t : \nabla E_t := \overline{E'_t \setminus E_t} \longrightarrow \bar{D}_t$$

is a differentiable fibration and trivial: $\nabla E_t \overset{\text{diffeo}}{\simeq}$ fibre $\times \bar{D}_t$, as fibrations over $\bar{D}_t$. We may take the typical fibre to be the one over $u \in \partial \bar{D}_t$:

$$\nabla F_t := \nabla E_t \cap f_t^{-1}(u) = F_0 \setminus \mathrm{int} F_t.$$

We use the following lemma which will be proved in §3.4.

3.3. **Lemma.** *Under the assumptions of Theorem 3.2 and with the notations above, we have:*

a) *If* $\dim X_t > 3$, *then the inclusion* $F_t \hookrightarrow F_0$ *is a homotopy equivalence (cf.* [21]*);*
b) *If* $\dim X_t > 3$, *then* $\pi_1(\partial F_0) \cong \pi_1(\nabla F_t) \cong \pi_1(\partial F_t)$ *(loc. cit.);*
c) $H_*(\partial F_t, \mathbb{Z}) \cong H_*(\nabla F_t, \mathbb{Z})$;
d) $H_*(\partial F_0, \mathbb{Z}) \cong H_*(\nabla F_t, \mathbb{Z})$.

All the isomorphisms are induced by appropriate inclusion maps.

In case $\dim X_t = 2$, the statements 1) and 2) follow from Theorem 2.10 and the fact that the Milnor fibres are bouquets of 1-spheres.

We now consider the case $\dim X_t > 3$. The statement 1) follows from a) in Lemma 3.3, and 2) follows from 1) and a theorem of Dold [5, (6.3)].

3) By Morse theory, $\pi_1(\partial F_0) = \pi_1(F_0)$ and $\pi_1(\partial F_t) = \pi_1(F_t)$. Hence ∂F_0 and ∂F_t are simply connected since F_0 and F_t are simply connected by the assumption. By Whitehead's theorem $\partial F_0 \hookrightarrow \nabla F_t$ and $\partial F_t \hookrightarrow \nabla F_t$ are homotopy equivalences. Since $\dim_{\mathbb{R}} \nabla F_t \geq 6$, by h-cobordism theorem (cf. [18, 22]), $\nabla F_t \overset{\text{diffeo}}{\simeq} [0,1] \times \partial F_t$. Since

$$f_t : \partial \bar{B}_0 \cap X_t \cap f_t^{-1}(\bar{D}_t) \longrightarrow \bar{D}_t$$

is trivial,

$$f_t : \partial \bar{B}_0 \cap X_t \cap f_t^{-1}(\partial \bar{D}_t) \longrightarrow \partial \bar{D}_t$$

is also trivial. Hence E'_t can be obtained from E_t by attaching $\partial \bar{D}_t \times$collar. This proves that $E'_t \overset{\text{diffeo}}{\simeq} E_t$ as fibrations over $\partial \bar{D}_t$.

4) Let $\Phi_t : E'_t \longrightarrow E_t$ and $\lambda_t : [0,1] \times \partial F_t \times \bar{D}_t \longrightarrow \nabla E_t = \nabla F_t \times \bar{D}_t$ be the diffeomorphisms of fibrations obtained in 3). Assume $\Phi_t(\lambda_t(0, z, u)) = \lambda_t(1, z, u)$. We also have a diffeomorphism of fibrations

$$\Psi_t : \partial \bar{B}_0 \cap X_t \cap f_t^{-1}(\bar{D}_t) \longrightarrow \partial \bar{B}_t \cap X_t \cap f_t^{-1}(\bar{D}_t)$$

and Φ_t and Ψ_t are equal at the points where both of them are defined. Furthermore

$$\lambda_t(0 \times \partial F_t \times 0) = \partial \bar{B}_0 \cap X_t \cap f_t^{-1}(0) \overset{\text{diffeo}}{\simeq} \lambda_t(1 \times \partial F_t \times 0) = \partial \bar{B}_t \cap X_t \cap f_t^{-1}(0)$$

under Ψ_t. By [15, Proposition 5.4] and its proof, there is a homeomorphism

$$\left[\bar{B}_t \cap X_t \cap f_t^{-1}(\partial \bar{D}_t)\right] \cup \left[\partial \bar{B}_t \cap X_t \cap f_t^{-1}(\bar{D}_t)\right] \longrightarrow \partial \bar{B}_t \cap X_t$$

preserving $\partial \bar{B}_t \cap f_t^{-1}(0)$. We have

$$\partial \bar{B}_t \cap X_t \cap f_t^{-1}(0) \overset{\text{homeo}}{\simeq} \partial \bar{B}_0 \cap X_t \cap f_t^{-1}(0) \overset{\text{homeo}}{\simeq} \partial \bar{B}_0 \cap X_0 \cap f_0^{-1}(0),$$

where the second homeomorphism comes from G_2. Hence

$$\begin{aligned} & \left(\bar{B}_t \cap X_t, \bar{B}_t \cap X_t \cap f_t^{-1}(0)\right) \\ \overset{\text{homeo}}{\simeq} \quad & \left(\bar{B}_t \cap X_t, \operatorname{Cone}(\partial \bar{B}_t \cap X_t \cap f_t^{-1}(0))\right) \\ \overset{\text{homeo}}{\simeq} \quad & \left(\bar{B}_0 \cap X_t, \operatorname{Cone}(\partial \bar{B}_0 \cap X_t \cap f_t^{-1}(0))\right) \\ \overset{\text{homeo}}{\simeq} \quad & \left(\bar{B}_0 \cap X_0, \operatorname{Cone}(\partial \bar{B}_0 \cap X_0 \cap f_0^{-1}(0))\right) \\ \overset{\text{homeo}}{\simeq} \quad & \left(\bar{B}_0 \cap X_0, \bar{B}_0 \cap X_0 \cap f_0^{-1}(0)\right), \end{aligned}$$

where the first and the last homeomorphisms were proved by Iomdin [9], the second and the third homeomorphisms follow from the above discussions.

Finally, returning to the case $\dim X_t = 2$, the proof of 3) and 4) follows from c) and d) of Lemma 3.3 together with the fact that a real two-dimensional homology cobordism is a product. ◇

3.4. **Proof of Lemma 3.3.** The proof of b) is essentially contained in the proof of Lemma 2.11.

a) In this special case, the map φ defined in §2.3 is in fact the inclusion map $F_t \hookrightarrow F_0$. It follows from Lemma 2.11 that the inclusion $F_t \hookrightarrow F_0$ induces an isomorphism of their fundamental groups. By Proposition 2.16, it also induces an isomorphism on the homology of the universal coverings of the spaces. Then we use Whitehead's theorem [28].

c) Since $H_*(F_0, F_t)$ is trivial, by using excision theorem we have $H_*(\nabla F_t, \partial F_t) = 0$. The proof of c) is finished.

d) The isomorphism in d) mainly comes from the Poincaré-Lefschetz duality theorem (see e.g. [23]). ◇

3.5. **Example.** Let X_t be the Briançon-Speder surfaces ([2], see also Example 2.9) defined by $h_t = z^3 + ty^{2\alpha+1}z + xy^{3\alpha+1} + x^{6\alpha+3} = 0$ $(\alpha \geq 1)$. These surfaces are quasi-homogeneous. Consider the function $f_t = xy^\alpha + z + tz^2$ on X_t. The critical locus of f_t is the solutions of the following system of equations:

$$\alpha(3z^2 + ty^{2\alpha+1})xy^{\alpha-1} - \left((2\alpha+1)ty^{2\alpha}z + (3\alpha+1)xy^{3\alpha}\right)(1+2tz) = 0 \tag{1}$$

$$(3z^2 + ty^{2\alpha+1})y^\alpha - \left((6\alpha+3)x^{6\alpha+2} + y^{3\alpha+1}\right)(1+2tz) = 0 \tag{2}$$

$$ty^{3\alpha}z + xy^{4\alpha} - 3\alpha x^{6\alpha+3}y^{\alpha-1} = 0 \tag{3}$$

$$xy^\alpha + z + 2tz^2 = 0 \tag{4}$$

$$h_t = 0 \tag{5}$$

The equations (1)–(3) come from the minors of the Jacobian of (f_t, h_t) and (4) comes from the differential of f_t by the Euler derivation.

Note that if any one of the x, y, or z is zero, then the other two are also zero. So we may assume $xyz \neq 0$. From (3), (4) and (5), one has

$$z = \omega x^{2\alpha+1}, \quad y^\alpha = -\omega u x^{2\alpha}, \quad 3\alpha x^2 = \omega^3 u^2 (t-u) y,$$

where $u := 1 + 2tz$ and $\omega^3 := -(3\alpha + 1)$. From this we obtain that $tz = c(t)$ with $c(0) \neq 0$. This means that the non-zero solutions of the above equations tend to infinity as t tends to 0. We conclude that, for small t, the singularity of (X_0, f_0) does not split in a neighborhood of the origin. By Theorem 3.2, the local topological type of f_t and the homotopy type of the Milnor fibration of f_t are constant.

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